Bound states of the s-wave Klein-Gordon equation with equal scalar and vector Standard Eckart Potential

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A supersymmetric technique for the bound-state solutions of the s-wave Klein–Gordon equation with equal scalar and vector standard Eckart type potential is proposed. Its exact solutions are obtained. Possible generalization of our approach is outlined.

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INTRODUCTION

Relativistic quantum mechanics is required to obtain more accurate results for the particle under a strong potential field. When we consider this condition, a particle in the strong potential field should be described by the Klein–Gordon equation and the Dirac equation.[1,2] In order to analyse relativistic effects on the spectrum of such a physical system, one may construct the Klein–Gordon equation including adequate potentials and obtains their solutions. In recent years, there have been many discussions about the Klein–Gordon equation with various types of potentials by using different methods to obtain the spectrum of the system. Some authors have considered the equality of scalar potential and vector potential in solving the Klein–Gordon equation as well as the Dirac equations for some potential fields.[3−7] The s-wave bound-state solutions are obtained in Refs. [5-8]. Similarly, the s-wave Klein–Gordon equation with vector potential and scalar Rosen-Morse type potentials[9,10] has been treated by the standard method,[11] the same problem with both the vector and scalar Hulthen-type potentials have been discussed analytically.[12] Energy spectrum of the s-wave Schrödinger equation with the generalized Hulthen potential has been obtained by using the supersymmetric quantum mechanics (SUSYQM) and supersymmetric Wentzel–Kramers–Brillouin (WKB) approach.[13] The bound-state spectra for some physical problems has been studied by the quantization condition and the SUSYQM.[14−16]

In this Letter, we construct a Klein–Gordon equation including the Eckart potential [17] whose spectrum can exactly be determined. For this purpose we transform the Klein–Gordon equation in the form of the Schrödinger-like equation, because there are many papers to tackle the problem in the framework of Schrödinger equations. The eigenvalues and eigenfunctions of the Eckart potential are obtained in terms of the SUSYQM.

SUSYQM APPROACH TO BOUND STATE SOLUTION

Generally, the s-wave Klein–Gordon equation with scalar potential $S(r)$ and vector potential $V(r)$ can be written [12,17] ($\hbar = 1, c = 1$)

$$\left\{\frac{d^2}{dr^2} + [E - V(r)]^2 - [M + S(r)]^2\right\} f(r) = 0, \tag{1}$$

where E is the energy, and M is the mass of the particle. Indeed, the original wavefunction can be expressed as $R(r) = f(r)/r$. We consider the standard Eckart potential in the form

$$V(r) = V_1 \mathrm{sech}^2(\alpha r) - V_2 \tanh(\alpha r). \tag{2}$$

When we consider the case that the vector potential and the scalar potential are equal, i.e. $V(r) = S(r)$, Eq. (1) becomes a well-known Schrödinger equation

$$\left\{\frac{d^2}{dr^2} + (E^2 - M^2) - 2(E + M)[V_1 \mathrm{sech}^2(\alpha r) - V_2 \tanh(\alpha r)]\right\} f(r) = 0, \tag{3}$$

with the effective potential

$$V_{\text{eff}}(r) = 2(E + M)[V_1 \mathrm{sech}^2(\alpha r) - V_2 \tanh(\alpha r)]. \tag{4}$$

Then Eq. (3) takes the form

$$\left\{-\frac{d^2}{dr^2}+V_{\rm eff}(r)\right\}f(r)=\lambda f(r), \tag{5}$$

where $\lambda = E^2 - M^2$ is the redefined energy parameter. In order to solve Eq. (5), in the framework of the SUSYQM, we introduce the following ground-state wave function

$$f_0(r)=N\exp\left[\int W(r)dr\right], \tag{6}$$

where N is a normalization constant, and $W(r)$ refers to a super-potential. Substituting Eq. (6) into Eq. (5), we obtain

$$W^2(r)-W'(r)=V_{\rm eff}(r)-\lambda_0, \tag{7}$$

where λ_0 is the ground-state energy, and Eq. (7) is a nonlinear Riccati equation which gives the wavefunction of the system.

Our task is now, to obtain the super potential $W(r)$, which is helpful to express the super partner potentials $V_+(r)$ and $V_-(r)$. After some straightforward calculation, we obtain the super potential $W(r)$, which can be written as

$$W(r)=A-B\tanh(\alpha r), \tag{8}$$

where A and B are the constant coefficients. Notice that the result Eq. (8) shows that the problem can be treated in the framework of the SUSYQM. The

super-symmetric partner potentials are given by

$$V_\pm(r)=W^2(r)\pm W'(r). \tag{9}$$

Substituting Eq. (8) into Eq. (9), we obtain the following partner potentials

$$V_+(r) = A^2+B^2-B(B+\alpha){\rm sech}^2(\alpha r)-2AB\tanh(\alpha r), \tag{10a}$$
$$V_-(r) = A^2+B^2-B(B-\alpha){\rm sech}^2(\alpha r)-2AB\tanh(\alpha r). \tag{10b}$$

In order to obtain λ_0 and the relations of A and B with V_1 and V_2, we compare Eqs. (3), (7), (10a),and (10b). As a consequence, one can easily obtain the following relations

$$\lambda_0 = A^2+B^2, \tag{11a}$$
$$B = \frac{1}{2}\left[\alpha\pm\sqrt{8(E+M)V_1+\alpha^2}\right], \tag{11b}$$
$$A = \frac{2(E+M)V_2}{\alpha\pm\sqrt{8(E+M)V_1+\alpha^2}}. \tag{11c}$$

It is well known that the potentials are shape invariant., that is, $V_+(r)$ has the same functional form as $V_-(r)$ but different parameters except for an additive constant. shape invariant condition can be expressed as

$$V_+(r,a_0)=V_-(r,a_1)+R(a_{1)}), \tag{12}$$

where a_0 and a_1 represents the potential parameters in the supersymmetric partner potentials, and $R(a_1)$ is a constant. This property permits an immediate analytical determination of eigenvalues and eigenfunctions. It is obvious that the potentials are invariant when the following conditions hold

$$\begin{aligned} a_0 &= -\frac{AB}{\alpha-B}, \\ a_1 &= -\alpha+B, \\ R(a_1) &= a_0^2+a_1^2-(A^2+B^2). \end{aligned}$$

Thus, the energy eigenvalues of Hamiltonian which includes $V_-(r)$ partner potential $-\frac{d^2}{dr^2}+V_-(r)$ are given by

$$\lambda_0^{(-)} = 0, \tag{13}$$
$$\lambda_n^{(-)} = \sum R(a_k)=\left(-\frac{AB}{\alpha n-B}\right)^2+(-\alpha n+B)^2-(A^2+B^2). \tag{14}$$

Therefore, the complete energy spectrum are obtained by

$$\lambda_n = \lambda_n^{(-)} + \lambda_0 = \left(-\frac{AB}{\alpha n - B}\right)^2 + (-\alpha n + B)^2, \quad n = 1, 2, 3, \cdots . \tag{15}$$

Substituting the values of coefficients A, B, and λ_0 into Eq. (15), we obtain the required relativistic bound-state energy spectrum

$$M^2 - E_n^2 = \frac{(E_n + M)^2 V_2^2}{\alpha^2} \frac{1}{(n+\delta)^2} + \alpha^2 (n+\delta)^2, \tag{16}$$

where the parameters δ is defined by $\delta = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8(E_n+M)V_1}{\alpha^2}}$.

The corresponding unnormalized ground-state wavefunction is determined by Eq. (6),

$$f_0(r) = N \exp(-Ar)(\cosh(\alpha r))^{B/\alpha}. \tag{17}$$

By using the parameters of A, B, and $f_0(r)$, we obtain the wavefunction in the form

$$R_0(r) = \frac{1}{r}[\cosh(\alpha r)]^{(p+w)} \exp(\alpha[w-p]r)), \tag{18}$$

where

$$p = \frac{1}{2}\left[n + \delta + \frac{2(E+M)V_2}{\alpha^2}\frac{1}{n+\delta}\right], \quad w = \frac{1}{2}\left[n + \delta - \frac{2(E+M)V_2}{\alpha^2}\frac{1}{n+\delta}\right].$$

The unnormalized wavefunction of the related Hamiltonian can be obtained by a similar mathematical procedure presented in Ref. [11]. By using this method, the wavefunction can be expressed in terms of the Jacobi polynomial as

$$R(r) = \frac{1}{r}[\cosh(\alpha r)]^{(p+w)} \exp[\alpha(w-p)r] \times P_n^{-2p,-2w}[-\coth(\alpha r)]. \tag{19}$$

This equation gives the required wavefunction of the standard Eckart potential with the Klein–Gordon equation.

CONCLUSIONS

In summary, we have discussed the exact solution of the Klein–Gordon equation including the equal scalar and vector Eckart potentials by using the SUSYQM. We have shown that both the eigenvalues and eigenfunctions of the Klein–Gordon equation can be obtained in the closed form for the Eckart potential. Finally, we emphasize that the method discussed here can be generated for other potentials.

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